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CALCULUS 2

integrals.theory.techniques.sequences

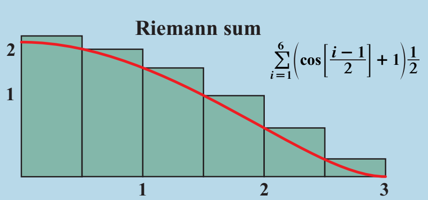
INTEGRAL & DIFFERENTIAL CALCULUS FOR ADVANCED STUDENTS

INTEGRATION

DEFINITIONS

- **Heuristics.** The definite integral captures the idea of adding the values of a function over a continuum.
- **Riemann sum.** A suitably weighted sum of values. A definite integral is the limiting value of such sums. A Riemann sum of a function f defined on $[a,b]$ is determined by a **partition**, which is a finite division of $[a,b]$ into subintervals, typically expressed by $a=x_0<x_1\cdots<x_n=b$; and a **sampling** of points, one point from each subinterval, say c_i from $[x_{i-1},x_i]$. The associated Riemann sum is: $\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})$.

A **regular partition** has subintervals all the same length, $\Delta x=(b-a)/n$, $x_i=a+i\Delta x$. A partition's **norm** is its maximum subinterval length. A **left sum** takes the left endpoint $c_i=x_{i-1}$ of each subinterval; a **right sum**, the right endpoint. An **upper sum** of a continuous f takes a point c_i in each subinterval where the maximum value of f is achieved; a **lower sum**, the minimum value. E.g., the upper Riemann sum of $\cos x$ on $[0,3]$ with a regular partition of n intervals is the left sum (since the cosine is decreasing on the interval): $\sum_{i=1}^{n}\left[\cos\left((i-1)\frac{3}{n}\right)+1\right]\frac{3}{n}$.



- **Definite integral.** The **definite integral** of f from a to b may be described as $\int_a^b f(x)dx=\lim_{\|\Delta x\|\to 0}\sum_i f(c_i)\Delta s_i$.

The limit is said to exist if some number S (to be called the integral) satisfies the following: Every $\varepsilon>0$ admits a δ such that all Riemann sums on partitions of $[a,b]$ with norm less than δ differ from S by less than ε. If there is such a value S, the function is said to be **integrable** and the value is denoted $\int_a^b f(x)dx$ or $\int_a^b f$. The function must be bounded to be integrable. The function f is called the **integrand** and the points a and b are called the **lower limit** and **upper limit** of integration, respectively. The word **integral** refers to the formation of $\int_a^b f$ from f and $[a,b]$, as well as to the resulting value if there is one.

- **Antiderivative.** An antiderivative of a function f is a function A whose derivative is f: $A'(x)=f(x)$ for all x in some domain (usually an interval). Any two antiderivatives of a function on an interval differ by a constant (a consequence of the **Mean Value Theorem**). E.g., both $\frac{1}{2}(x-a)^2$ and $\frac{1}{2}x^2-ax$ are antiderivatives of $x-a$, differing by $\frac{1}{2}a^2$. The **indefinite integral** of a function f, denoted $\int f(x)dx$, is an expression for the family of antiderivatives on a typical (often unspecified) interval. E.g., (for $x<-1$, or for $x>1$).

$$\int\frac{x}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+C.$$

The constant C, which may have any real value, is the **constant of integration**. (Computer programs, and this chart, may omit the constant, it being understood by the knowledgeable user that the given antiderivative is just one representative of a family.)

INTERPRETATIONS

- **Area under a curve.** If f is nonnegative and continuous on $[a,b]$, then $\int_b^a f(x)dx$ gives the area between the x-axis and the graph. The **area function** $A(x)=\int_a^x f(t)dt$ gives the area accumulated up to x. If f is negative, the integral is the negative of the area.
- **Average value.** The average value of f over an interval $[a,b]$ may be defined by ***average value*** $=\frac{1}{b-a}\int_a^b f(x)dx$.

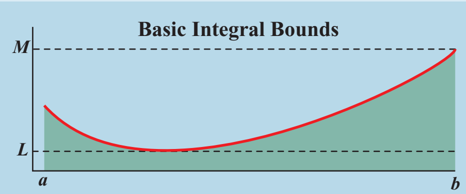
A rough estimate of an integral may be made by estimating the average value (by inspecting the graph) and multiplying it by the length of the interval. (See *Mean Value Theorem (MVT) for integrals,* in the *Theory* section.)

- **Accumulated change.** The integral of a rate of change of a quantity over a time interval gives the total change in the quantity over the time interval. E.g., if $v(t)=s'(t)$ is a velocity (the rate of change of position), then $v(t)\Delta t$ is the approximate displacement occurring in the time increment t to $t+\Delta t$; adding the displacements for all time increments gives the approximate change in position over the entire time interval. In the limit of small time increments, one gets the exact total displacement: $\int_a^b v(t)dt=s(b)-s(a)$.
- **Integral curve.** Imagine that a function f determines a slope $f(x)$ for each x. Placing line segments with slope $f(x)$ at points (x,y) for various y, and doing this for various x, one gets a **slope field**. An antiderivative of f is a function whose graph is tangent to the slope field at each point. The graph of the antiderivative is called an **integral curve** of the slope field.
- **Solution to initial value problem.** The solution to the differential equation $y'=f(x)$ with initial value $y(x_0)=y_0$ is $(x)=y_0+\int_{x_0}^x f(t)dt$.

THEORY

- **Integrability & inequalities.** A continuous function on a closed interval is integrable. Integrability on $[a,b]$ implies integrability on closed subintervals of $[a,b]$. Assuming f is integrable, if $L\le f(x)\le M$ for all x in $[a,b]$, then $L\cdot(b-a)\le\int_a^b f(x)dx\le M\cdot(b-a)$.

Use this to check integral evaluations with rough overestimates or underestimates.

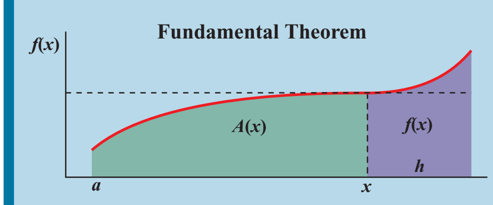


If f is nonnegative, then $\int_a^b f(x)dx$ is nonnegative.

If f is integrable on $[a,b]$, then so is f, and $\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx$.

- **Fundamental theorem of calculus.** One part of the theorem is used to **evaluate integrals**: If f is continuous on $[a,b]$, and A is an antiderivative of f on that interval, then $\int_a^b f(x)dx=A(x)\Big|_a^b\equiv A(b)-A(a)$.

The other part is used to **construct antiderivatives**: If f is continuous on $[a,b]$, then the function $A(x)|=\int_a^x f(t)dt$ is an antiderivative of f on $[a,b]$: $A'(x)=f(x)$ (valid for one-sided derivatives at the endpoints).



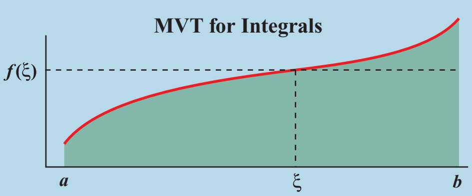
- **Differentiation of integrals.** Functions are often defined as integrals. E.g., the "sine integral function" is $Si(x)=\int_0^x\left(\frac{\sin t}{t}\right)dt$.

To differentiate such, use the second part of the fundamental theorem: $Si'(x)=\sin x/x$. A function such as $\int_a^{x^2} f(t)dt$ is a composition involving $A(u)=\int_a^u f(t)dt$. To differentiate, use the chain rule and the fundamental theorem:

$\frac{d}{dx}\int_a^{x^2} f(t)dt=\frac{d}{dx}A(x^2)=A'(x^2)2x=2xf(x^2)$.

- **Mean value theorem for integrals.** If f and g are continuous on $[a,b]$, then there is a ξ in $[a,b]$ such that $\int_a^b f(x)g(x)dx=f(\xi)\int_a^b g(x)dx$.

In the case $g\equiv 1$, the average value of f is attained somewhere on the interval: $\frac{1}{b-a}\int_a^b f(x)dx=f(\xi)$.



- **Change of variable formula.** An integrand and limits of integration can be changed to make an integral easier to apprehend or evaluate. In effect, the "area" is smoothly redistributed without changing the integral's value. If g is a function with continuous derivative and f is continuous, then $\int_a^b f(u)du=\int_c^d f(g(t))g'(t)dt$, where c,d are points with $g(c)=a$ and $g(d)=b$.

In practice, **substitute** $u=g(t)$; compute $du=g'(t)dt$; and find what t is when $u=a$ and $u=b$. E.g., $u=\sin t$ effects the transformation $\int_a^b\sqrt{1-u^2}\,du=\int_0^{\pi/2}\sqrt{1-\sin^2 t}\cos t\,dt$, which becomes $\int_0^{\pi/2}\cos^2 t\,dt$, since $\sqrt{1-\sin^2 t}=\cos t$ for $0\le t\le\pi/2$. The formula is often used in reverse, starting with $\int_b^a F(g(x))g'(x)dx$. See *Techniques* on pg. 2.

- **Natural logarithm.** A rigorous definition is $\ln x=\int_1^x\frac{1}{u}du$. The change of variable formula with $u=1/t$ yields $\int_1^{1/x}\frac{1}{u}du=\int_1^x t\frac{-1}{t^2}dt=-\int_1^x\frac{1}{t}dt$ showing that $\ln(1/x)=-\ln x$. The other elementary properties of the natural log can likewise be easily derived from this definition. In this approach, an inverse function is deduced and is defined to be the natural exponential function.

INTEGRATION FORMULAS

- **Basic indefinite integrals.** Each formula gives just one antiderivative (all others differing by a constant from that given), and is valid on any open interval where the integrand is defined:

$\int x^n\,dx=\frac{x^{n+1}}{n+1}(n\neq-1)$ $\int\frac{1}{x}dx=\ln|x|$

$\int e^{kx}\,dx=\frac{e^{kx}}{k}(k\neq 0)$ $\int a^x\,dx=\frac{a^n}{\ln a}(a\neq 1)$

$\int\cos x\,dx=\sin x$ $\int\sin x\,dx=-\cos x$

$\int\frac{dx}{1+x^2}=\arctan x$ $\int\frac{dx}{\sqrt{1-x^2}}=\arcsin x$

- **Further indefinite integrals.** The above conventions hold:

$\int\tan x\,dx=\ln|\sec x|$ $\int\cot x\,dx=\ln|\sin x|$

$\int\sec x\,dx=\ln|\sec x+\tan x|$

$\int\csc x\,dx=\ln|\csc x+\cot x|$

$\int\cosh x\,dx=\sinh x$ $\int\sinh x\,dx=\cosh x$

$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ $\int|x|dx=\frac{1}{2}x|x|$

$\int\frac{dx}{\sqrt{x^2+a^2}}=\ln\left|x+\sqrt{x^2+a^2}\right|=\sinh^{-1}\frac{x}{a}+\ln a$

$\int\frac{dx}{\sqrt{x^2a^2}}=\ln\left|x+\sqrt{x^2a^2}\right|\cosh^{-1}\frac{x}{a}+\ln a$

(take positive values for **cosh-1**)

$\int\sqrt{x^2\pm a^2}\,dx=\frac{1}{2}x\sqrt{x^2a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|$

(Take same sign, + or –, throughout)

$\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}$

- **Common definite integrals:**

$\int_0^1 x^n\,dx=\frac{1}{n+1}$ $\int_0^r\sqrt{r^2-x^2}\,dx=\frac{\pi r^2}{4}$ $\int_0^\pi\sin x\,dx=2$

$\int_0^{\pi/2}\cos^2\theta\,d\theta=\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta=\frac{\pi}{4}$

$\int_0^{\pi/2}\sin^2\theta\,d\theta=\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}d\theta=\frac{\pi}{4}$

To remember which of $^1/_2\,(1\pm\cos 2\theta)$ equals $\cos^2\theta$ or $\sin^2\theta$, recall the value at zero.

TECHNIQUES

- **Substitution.** Refers to the **Change of variable formula** (see the *Theory* section), but often the formula is used in reverse. For an integral recognized to have the form $\int_a^b F(g(x))g'(x)dx$ (with F and g' continuous), you can put $u=g(x)$, $du=g'(x)dx$, and modify the limits of integration appropriately: $\int_a^b F(g(x))g'(x)dx=\int_{g(a)}^{g(b)}F(u)du$.

In effect, the integral is over a path on the u-axis traced out by the function g. (If $g(b)=g(a)$ [the path returns to its start], then the integral is zero.) E.g., $u=1+x^2$ yields $\int_0^1=\frac{x}{1+x^2}dx=\frac{1}{2}\int_0^1\frac{1}{1+x^2}2\,x\,dx=\frac{1}{2}\ln(1+x^2)$.

Substitution may be used for indefinite integrals.

E.g., $\int\frac{x}{1+x^2}dx=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln u=\frac{1}{2}\ln(1+x^2)$.

Some **general formulas** are:

$\int g(x)^n g'(x)dx=\frac{g(x)^{n+1}}{n+1}, \int\frac{g'(x)}{g(x)}dx=\ln|g(x)|,$

$\int e^{g(x)}g'(x)dx=e^{g(x)}.$

- **Integration by parts.** Explicitly,

$\int_a^b u(x)v'(x)dx=u(x)v(x)\big|_a^b-\int_a^b v(x)u'(x)dx.$

The common formula is $\int_a^b u\,dv=uv\big|_a^b-\int_a^b v\,du.$

For indefinite integration, $\int u\,dv=uv-\int v\,du.$

The procedure is used in derivations where the functions are general, as well as in explicit integrations. You don't need to use "u" and "v." View the integrand as a product with one factor to be integrated and the other to be differentiated; the integral is the integrated factor times the one to be differentiated, minus the integral of the product of the two new quantities. The factor to be integrated may be **1** (giving $v=x$).

E.g., $\int\arctan x\,dx=x\arctan x-\int\frac{x}{1+x^2}dx$

Other routine integration-by-parts integrands are $\arcsin x$, $\ln x$, $x^n\ln x$, $x\sin x$, $x\cos x$, and xe^{ax}.

- **Rational functions**. Every rational function may be written as a polynomial plus a proper rational function (degree of numerator less than degree of denominator). A proper rational function with real coefficients has a **partial fraction decomposition**: It can be written as a sum with each summand being either a constant over a power of a linear polynomial or a linear polynomial over a power of a quadratic. A factor $(x-c)^k$ in the denominator of the rational function implies there could be summands $\frac{A_1}{x-c}+\ldots+\frac{A_k}{(x-c)^k}$.

A factor $(x^2+bx+c)^k$ (the quadratic not having real roots) in the denominator implies there could be summands $\frac{A_1+B_1x}{x^2+bx+c}+\ldots+\frac{A_k+B_kx}{(x^2+bx+c)^k}$.

Math software can handle the work, but the following case should be familiar. If $a\neq b$, $\frac{1}{(x-a)(x-b)}=\frac{C}{x-a}+\frac{D}{x-b}$

where C, D are seen to be $C=-D=\frac{1}{a-b}$.

Thus $\int\frac{1}{(x-a)(x-b)}dx=\frac{1}{a-b}(\ln|x-a|-\ln|x-b|)$.

In general, the indefinite integral of a proper rational function can be broken down via partial fraction decomposition and linear substitutions (of form $u=ax+b$) into the integrals $\int u^{-1}du, \int u^{-n}du\,(n>1), \int u(u^2+1)^{-n}du$ (handled with substitution $w=u^2+1$), and $\int(u^2+1)^{-n}du$ (handled with substitution $u=\tan t$).

IMPROPER INTEGRALS

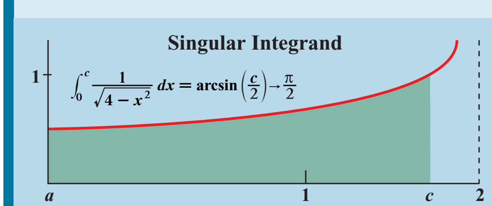
- **Unbounded limits.** If f is defined on $[a,\infty]$ and integrable on $[a,B]$ for all $B>a$, then $\int_a^\infty f(x)dx\overset{def}{=}\lim_{B\to\infty}\int_a^B f(x)dx$ provided the limit exists.

E.g., $\int_0^\infty e^{-x}dx=\lim_{B\to\infty}(1-e^{-B})=1$

Likewise, for appropriate f, $\int_{-\infty}^b f(x)dx\overset{def}{=}\lim_{A\to\infty}\int_A^b f(x)dx.$

In each case, if the limit exists, the improper integral **converges**, and otherwise it **diverges**. For f defined on $(-\infty,\infty)$ and integrable on every bounded interval, $\int_{-\infty}^\infty f(x)dx\overset{def}{=}\lim_{A\to\infty}\int_A^c f(x)dx+\lim_{B\to\infty}\int_c^B f(x)dx$ (the choice of c being arbitrary), provided each integral on the right converges.

- **Singular integrands**. If f is defined on $(a,b]$ but not at $x=a$ and is integrable on closed subintervals of $(a,b]$, then $\int_a^b f(x)dx\overset{def}{=}\lim_{c\to a^+}\int_c^b f(x)dx$ provided the limit exists. A similar definition holds if the integrand is defined on $[a,b)$. E.g., $\int_0^2\frac{1}{\sqrt{4-x^2}}dx$ is $\lim_{c\to 2^-}\int_0^c\frac{1}{\sqrt{4-x^2}}dx=\lim_{c\to 2^-}\arcsin\left(\frac{c}{2}\right)=\frac{\pi}{2}$.



If f is not defined at a finite number of points in an interval $[a,b]$, and is integrable on closed subintervals of open intervals between such points, the integral $\int_a^b f$ is defined as a sum of left and right-hand limits of integrals over appropriate closed subintervals, provided all the limits exist.

E.g., $\int_{-1}^1\frac{1}{x^3}dx=\lim_{a\to 0^-}\int_{-1}^a\frac{1}{x^3}dx+\lim_{b\to 0^+}\int_b^1\frac{1}{x^3}dx$ if the limits on the right were to exist. They don't, so the integral diverges.

- **Examples & bounds**.

$\int_1^\infty\frac{1}{x^p}dx$ converges for $p>1$, diverges otherwise.

$\int_0^1\frac{1}{x^p}dx$ converges for $p<1$, diverges otherwise.

$\int_2^\infty\frac{1}{x(\ln x)^p}dx$ converges for $p>1$, diverges otherwise.

Note: $\int\frac{dx}{x(\ln x)^p}=-\frac{1}{(n-1)(\ln x)^{p-1}}$, $p>1$ converges at $x=\infty$, $p=0$ or <1 diverges at $x=\infty$.

E.g., $\int_1^\infty\frac{1}{x^2}dx$ converges to **1** and $\int_0^1\frac{1}{x^2}dx$ diverges.

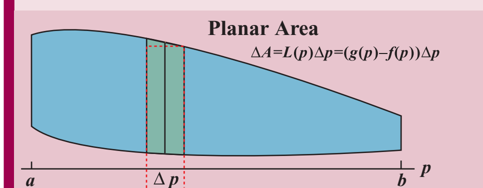
The above integrals are useful in comparisons to establish convergence (or divergence) and to get bounds.

E.g., $\int_0^\infty\left(\frac{x}{1+x^2}\right)^{3/2}dx$ converges since the integrand is bounded by $1/2^{3/2}\}$ on $[0,1]$ and is always less than $1/x^{3/2}$. It converges to a number less than $\frac{1}{2^{3/2}}\int_1^\infty\frac{1}{x^{3/2}}dx=\frac{1}{2^{3/2}}+2<2.4$.

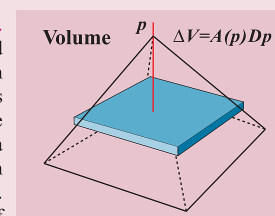
APPLICATIONS

GEOMETRY

- **Areas of plane regions.** Consider a plane region admitting an axis such that sections perpendicular to the axis vary in length according to a known function $L(p)$, $a\leq p\leq b$. The area of a strip of width Δp perpendicular to the axis at p is $\Delta A=L(p)\Delta p$, and the total area is $A=\int_a^b L(p)dp$. E.g., the area of the region bounded by the graphs of f and g over $[a,b]$ is $\int_a^b[g(x)-f(x)]dx$, provided $g(x)\geq f(x)$ on $[a,b]$. Sometimes it is simpler to view a region as bounded by two graphs "over" the y-axis, in which case the integration variable is y.



- **Volumes of solids.** Consider a solid admitting an axis such that cross-sections perpendicular to the axis vary in area according to a known function $A(p)$, $a\leq p\leq b$.



The volume of a slab of thickness Δp perpendicular to the axis at p is $\Delta V=A(p)\Delta p$, and the total volume is $V=\int_a^b A(p)dp$. E.g., a pyramid having square horizontal cross-sections, with bottom side length s and height h, has cross-sectional area $A(z)=[s(1-z/h)]^2$ at height z. Its volume is thus $V=\int_0^h s^2\left(1-\frac{z}{h}\right)^2dz=\frac{s^2h}{3}$.

- **Solids of revolution**. Consider a solid of revolution determined by a known radius function $r(z)$, $a\leq z\leq b$, along its axis of revolution. The area of the cross-sectional "**disk**" at z is $A(z)=\pi r(z)^2$, and the volume is $V=\int_a^b A(z)dz=\int_a^b\pi r(z)^2dz$.

If the solid lies between two radii $r_1(z)$ and $r_2(z)$ at each point z along the axis of revolution, the cross-sections are "**washers**," and the volume is the obvious difference of volumes like that above. Sometimes a radial coordinate r, $a\leq r\leq b$, along an axis perpendicular to the axis of revolution, parametrizes the heights $h(r)$ of cylindrical sections (**shells**) of the solid parallel to the axis of revolution. In this case, the area of the shell at r is $A(r)=2\pi r\,h(r)$, and the volume of the solid is $V=\int_a^b A(r)dr=\int_a^b 2\pi rh(r)dr$.

- **Arc length.** A graph $y=f(x)$ between $x=a$ and $x=b$ has length $V=\int_a^b\sqrt{1+f'(x)^2}\,dx$. A curve C parametrized by $((x(t), y(t))$, $a\le t\le b$, has length $\int_C ds=\int_a^b\sqrt{x'(t)^2+y'd(t)^2}\,dt$.
- **Area of a surface of revolution.** The surface generated by revolving a graph $y=f(x)$ between $x=a$ and $x=b$ about the x-axis has area $\int_b^a 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$. If the generating curve C is parametrized by $((x(t), y(t))$, $a\le t\le b$, and is revolved about the x axis, the area is $\int_C 2\pi y\,ds=\int_a^b 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2}\,dt$.

PHYSICS

- **Motion in one dimension.** Suppose a variable displacement $x(t)$ along a line has velocity $v(t)=x'(t)$ and acceleration $a(t)=v'(t)$. Since v is an antiderivative of a, the fundamental theorem implies: $v(t) = v(t_0) + \int_{t_0}^t a(u)du$, $x(t)=x(t_0)+\int_{t_0}^t v(u)du$. E.g., the height $x(t)$ of an object thrown at time $t_0=0$ from a height $x(0)=x_0$ with a vertical velocity $v(0)=v_0$ undergoes the acceleration $-g$ due to gravity. Thus $v(t) = v(v_0) + \int_0^t(-u)du = v_0-gt$ and $x(t) = x_0 + \int_0(v_0-gu)du = x_0 + v_0t-\frac{1}{2}gt^2$.
- **Work.** If $F(x)$ is a variable force acting along a line parametrized by x, the approximate work done over a small displacement Δx at x is $\Delta W=F(x)\Delta x$ (force times displacement), and the work done over an interval $[a,b]$ is $W=\int_a^b F(x)dx$.

  In a **fluid lifting** problem, often $\Delta W=\Delta F\bullet h(y)$, where $h(y)$ is the lifting height for the "**slab**" of fluid at y with cross-sectional area $A(y)$ and width Δy, and the slab's weight is $\Delta F=\rho A(y)\Delta y$, ρ being the fluid's weight-density. Then $W=\int_a^b \rho A(y)h(y)dy$.

DIFFERENTIAL EQUATIONS

- **Examples.** A differential equation (DE) was solved in the item **Solution to initial value problem**; an example of that type is in **Motion in one dimension**. In those, the expression for the derivative involved only the independent variable. A basic DE involving the dependent variable is $y'=ky$. A general DE where only the **first–order** derivative appears and is **linear** in the dependent variable is $y'+p(t)y=q(t)$.

  Generally more difficult are equations in which the independent variable appears in a \hlt{nonlinear} way; e.g., $y'=y^2-x$. Common in applications are **second-order** DEs that are linear in the dependent variable; e.g., $y''=-ky$, $x^2y''+xy'+x^2y=0$.
- **Solutions.** A solution of a DE on an interval is a function that is differentiable to the order of the DE and satisfies the equation on the interval. It is a **general solution** if it describes virtually all solutions, if not all. A general solution to an nth order DE generally involves n constants, each admitting a range of real values. An **initial value problem** (IVP) for an nth order DE includes a specification of the solution's value and n–1 derivatives at some point. Generally in applications, an IVP has a **unique solution** on some interval containing the initial value point.
- **Basic first-order linear DE.** The equation $y'=ky$, rewritten $\frac{dy}{dt}=ky$ suggests $\frac{dy}{y}=kdt$ where $|y|=kt+c$. In this way, one finds a solution $y=Ce^{kt}$. On any open interval, every solution must have that form, because $y'=ky$ implies $\frac{d}{dt}(ye^{-kt})$, where ye^{-kt} is constant on the interval. Thus $y=Ce^{kt}$ (C real) is the general solution. The unique solution with $y(a)=y_a$ is $y=y_ae^{k(t-a)}$. The **trivial solution** is $y\equiv 0$, solving any IVP $y(a)=0$.
  - **General first-order linear DE.** Consider $y'+p(t)y=q(t)$. The solution to the associated **homogeneous equation** $h'+p(t)h=0$ $(dh/h=-p(t)dt)$ with $h(a)=1$ is $h(t)=\exp\left[-\int_a^t p(u)du\right]$.

  If y is a solution to the original DE, then $(y/h)'=q/h$, where $y=h\int q/h$. The solution with $y(a)=y_a$ is $y(t)=Y_a+\left[-\int_a^t q(u)h(u)^{-1}du\right]$.

APPROXIMATIONS

TAYLOR'S FORMULA

- **Taylor polynomials.** The nth degree Taylor polynomial of f at c is $P_n(x) = f(c) + f'(c)(x-c) + \frac{1}{2}!f''(c)(x-c)^2 + ...+ \frac{1}{n!}f^{(n)}(c)(x-c)^n$ (provided the derivatives exist). When $c=0$, it's also called a **MacLaurin polynomial**.
- **Taylor's formula.** Assume f has n+1 continuous derivatives on open interval and that c is a point in the interval. Then for any x in the interval, $f(x)=P_n(x)+R_n(x)$, where $R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)\bullet(x-c)^{n+1}$ for some ξ between c and x (ξ varying with x). The expression for $R_n(x)$ is called the **Lagrange form of the remainder**. E.g., the remainders for the MacLaurin polynomials of $f(x)$ = $\ln(1+x)$, $-1<x<1$, are $R_n(x) = \frac{(-1)^n}{(n+1)(1+\xi)^{n+1}}\bullet x^{n+1}$.

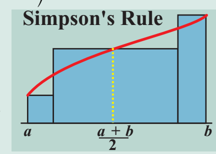
  There is a ξ between 0 and x such that $\ln(1 + x) = x-\frac{x^2}{2}+\frac{1}{3(1+\xi)^3}$.
- **Error bounds.** As x approaches c, the remainder generally becomes smaller, and a given Taylor polynomial provides a better approximation of the function value. With the assumptions and notation above, if $\left|f^{(n+1)}(x)\right|$ is bounded by M on the interval, then $|f(x)-P_n(x)| \le\frac{M}{(n+1)!}|x-c|^{n+1}$ for all x in the interval. E.g., for $|x|<1$, $e^x\approx 1+x+x^2/2$, with error no more than $\frac{3}{3!}|x|^3=0.5|x|^3$, because the third derivative of e^x is bounded by 3 on $(-1,1)$.
- **Big O notation.** The statement $f(x)=p(x)+O(x^m)$ (as $x\to 0$) means that $\frac{f(x)-p(x)}{x^m}$ is bounded near $x=0$. (Some authors require that the limit of this ratio as x approaches 0 exist.) That is, $f(x)-p(x)$ approaches 0 at essentially the same rate as x^m. E.g., Taylor's formula implies $f(x)=f(0)\neq f'(0)x+\frac{1}{2}f''(0)x^2+O(x^3)$ if f has continuous third derivative on an open interval containing 0. E.g., $\sin x=x+O(x^3)$. [Similar relations can be inferred from the identities in the item *Basic MacLaurin Series.*]
- **L'Hôpital's rule.** This resolves indeterminate ratios or $\left(\frac{0}{0}\text{or}\frac{\infty}{\infty}\right)$. If $\lim_{x\to a}f(x) = 0 = \lim_{x\to a}g(x)$ and if $\lim_{x\to a}{}'f(x) = 0 = \lim_{x\to a}{}'g(x)$ are defined and $g(x)\neq 0$, for x near a (but not necessarily at a), then $\lim_{x\to a}\frac{f(x)}{g(x)}=0=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided the latter limit exists, or is infinite. The rule also holds when the limits of f and g are infinite. Note that $f'(a)$ and $g'(a)$ are not required to exist. To resolve an indeterminate difference $(\infty-\infty)$, try to rewrite it as an indeterminate ratio and apply L'Hôpital's rule. To resolve an indeterminate exponential $(0^0, 1^\infty, \text{or } \infty^0)$, take its logarithm to get a product and rewrite this as a suitable indeterminate ratio; apply L'Hôpital's rule; the exponential of the result resolves the original indeterminate exponential.

  For $\lim_{x\to 0}|x|^x$ you get and find $\lim_{x\to 0}\frac{\ln|x|}{1/x}=\lim_{x\to 0}\frac{1/x}{-1/x^2}=0$, where $\lim_{x\to 0}|x|^x = e^0 = 1$.

NUMERICAL INTEGRATION

- **General notes.** Solutions to applied problems often involve definite integrals that cannot be evaluated easily, if at all, by finding antiderivatives. Readily available software using refined algorithms can evaluate many integrals to needed precision. The following methods for approximating $\int_b^a f(x)dx$ are elementary. Throughout, n is the number of intervals in the underlying regular partition and $h=(b-a)/n$.
- **Trapezoid rule.** The line connecting two points on the graph of a positive function together with the underlying interval on the x axis form a trapezoid whose area is the average of the two function values times the length of the interval. Adding these areas up over a regular partition gives the trapezoid rule approximation

  $T_n=\left(\frac{f(a)}{2}+\sum_{i=1}^{n-1}f(a-ih)+\frac{f(b)}{2}\right)h$. This is also the average of the left sum and right sum for the given partition. The approximation remains valid if f is not positive.
- **Midpoint rule.** This evaluates the Riemann sum on a regular partition with the sampling given by the midpoints of each interval: $M_n=\sum_{I=0}^{n-1}f\left(a+\left[i-\frac{1}{2}\right]h\right)h$. Each summand is the area of a trapezoid whose top is the tangent line segment through the midpoint.
- **Simpson's rule.** The weighted sum $\frac{1}{3}T_1+\frac{2}{3}M_1$ on the interval $[a,b]$ yields Simpson's rule $S=\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$.

  This is also the integral of the quadratic that interpolates the function at the three points. For a regular partition of $[a,b]$ into an even number $n=2m$ of intervals, a formula is:

  $S_{2m}=\frac{h}{3}\{f(a)+4\sum_{i=0}^{m-1}f(a)+[2i+1]h+2\sum_{i=0}^{m-1}f(a+2i\bullet h)+f(b)\}$

  where $h = (b - a)/n$. Simpson's rule is exact on cubics.

SEQUENCES & SERIES

SEQUENCES

- **Sequences.** Sequences are functions whose domains consist of all integers greater than or equal to some initial integer, usually 0 or 1. The integer in a sequence at n is usually denoted with a subscripted symbol like a_n (rather than with a functional notation $a(n)$) and is called a **term** of the sequence. A sequence is often referred to with an expression for its terms, e.g., $1/n$ (with the domain understood), in lieu of a fuller notation like: $\{1/n\}_{n=1}^{\infty}$, or $n|\to 1/n(n=1, 2, \ldots)$.
- **Elementary sequences.** An **arithmetic sequence** a_n has a common difference d between successive values: $a_n=a_{n-1}+d=a_0+d\cdot n$. It is a sequential version of a linear function, the common difference in the role of slope. A **geometric sequence,** with terms g_n, has a common ratio r between successive values: $g_n=g_{n-1}r=g_0r^n$. E.g., 5.0, 2.5, 1.25, 0.625, 0.3125,.... It is a sequential version of an exponential function, the common ratio in the role of base.
- **Convergence.** A sequence $\{a_n\}$ **converges** if some number L (called the limit) satisfies the following: Every $\varepsilon>0$ admits an N such that $|a_n-L|<\varepsilon$ for all $n\ge N$. If a limit L exists, there is only one; one says $\{a_n\}$converges to L, and writes $a_n\to L$, or $\lim_{n\to\infty}a_n=L$. If a sequence does not converge, it **diverges**. If a sequence a_n diverges in such a way that every $M>0$ admits an N such that $a_n>M$ for all $n\ge N$, then one writes $a_n\to\infty$. E.g., if $|r|<1$ then $r^n\to 0$; if $r=1$ then $r^n\to 1$; otherwise r_n diverges, and if $r>1$, $r^n\to\infty$.
- **Bounded monotone sequences.** An increasing sequence that is bounded above converges (to a limit less than or equal to any bound). This is a fundamental fact about the real numbers, and is basic to series convergence tests.

SERIES OF REAL NUMBERS

- **Series.** A series is a sequence obtained by adding the values of another sequence $\sum_{n=0}^{N}a_n=a_0+...+a_N$. The value of the series at N is the sum of values up to a_N and is called a **partial sum**: $\sum_{n=0}^{N}a_n=a_0+...+a_N$. The series itself is denoted $\sum_{n=0}^{\infty}a_n$. The a_n are called the **terms** of the series.
- **Convergence.** A series $\sum_{n=0}a_n$ **converges** if the sequence of partial sums converges, in which case the limit of the sequence of partial sums is called the **sum** of the series. If the series converges, the notation for the series itself stands also for its sum: $\sum_{n=0}^{\infty}a_n=\lim_{N\to\infty}\sum_{n=0}^{N}a_n$.

An equation such as $\sum_{n=0}^{\infty} a_n = S$ means the series converges and its sum is S. In general statements, $\sum a_n$ may stand for $\sum_{n=0}^{\infty} a_n = S$.

- **Geometric series**. A (numerical) geometric series has the form $\sum_{n=0}^{\infty} ar^n$, where r is a real number and $a \neq 0$. A key identity is $\sum_{n=0}^{N} r^n = 1+r+r^2+\ldots+r^N = \frac{1-r^{N+1}}{1-r} (r\neq 1)$. It implies $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ (if $|r|<1$) $\left(\text{also } \sum_{n=1}^{\infty} ar^n = a\left(\frac{1}{1-r}-1\right)\right)$, and that the series diverges if $|r|>1$. The series diverges if $r=\pm 1$. The convergence and possible sum of any geometric series can be determined using the preceding formula.

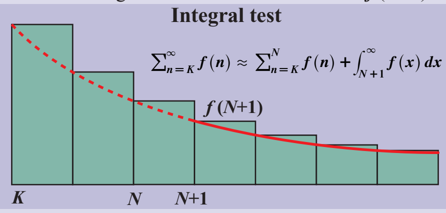
  E.g., $\sum_{n=1}^{\infty} \frac{4}{3^n} = 4\left(\frac{1}{1-\frac{1}{3}}-1\right) = 2.$

- **p-series**. For p, a real number, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series. The p-series diverges if $p \le 1$ and converges if $p>1$ (by comparison with harmonic series and the integral test, below). The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, for the partial sums are unbounded: $\sum_{n=1}^{2^N} \frac{1}{n} \ge 1+\frac{N}{2}$.
- **Alternating series.** These are series whose terms alternate in (nonzero) sign. If the terms of an alternating series strictly decrease in absolute value and approach a limit of zero, then the series converges. Moreover, the truncation error is less than the absolute value of the first omitted term: $\left|\sum_{n=1}^{\infty} (-1)^n a_n - \sum_{n=1}^{N} (-1)^n a_n\right| < a_{N+1}$. (assuming $a_n \to 0$ in a strictly decreasing manner).

CONVERGENCE TESTS

- **Basic considerations**. For any K, if $\sum_{n=K}^{\infty} a^n$ converges, then $\sum_{n=1}^{\infty} a^n$ converges, and conversely. If $a_n \not\to 0$, then $\sum a_n$ diverges. (Equivalently, if $\sum a_n$ converges, then $a_n \to 0$). This says nothing about, e.g., $\sum_{n=K}^{\infty} \frac{1}{n}$. A series of positive terms is an increasing sequence of partial sums; if the sequence of partial sums is bounded, the series converges. This is the foundation of all the following criteria for convergence.
- **Integral test & estimate**. Assume f is continuous, positive, and decreasing on (K,∞). Then $\sum_{n=K}^{\infty} f(n)$ converges if and only if $\int_K^{\infty} f(x)dx$, converges. If the series converges, then $\sum_{n=K}^{\infty} f(n) \le \sum_{n=K}^{N} f(n) + \int_N^{\infty} f(x)dx$, the right side overestimating the sum with error less than $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \sum_{n=1}^{12} \frac{1}{n^3} + \int_{13}^{\infty} \frac{1}{x^3} dx = 1.2018\ldots$, the left side underestimating the sum with error less than $f(N+1)$.

Integral test



E.g., $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \sum_{n=1}^{12} \frac{1}{n^3} + \int_{13}^{\infty} \frac{1}{x^3} dx = 1.2018\ldots$, an underestimate with error $<13^{-3}<5\cdot 10^{-4}$.

- **Absolute convergence**. If $\sum |a_n|$ converges, that is, if $\sum a_n$ {**converges absolutely**}, then $\sum a_n$ converges, and $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$. A series **converges conditionally** if it converges, but not absolutely.
- **Comparison tests**. Assume $a_n, b_n > 0$.
  - If $\sum b_n$ converges and either a_n£b_n $(n \ge N)$ or a_n/b_n has a limit, then $\sum a_n$ converges.
  - If $\sum b_n$ diverges and either b_n£a_n $(n \ge N)$ or a_n/b_n has a *nonzero* limit (or approaches ∞), then $\sum a_n$ diverges.

  The p-series and geometric series are often used for comparisons. Try a "limit" comparison when a series looks like a p-series, but is not directly comparable to it.

  E.g., $\sum_{n=K}^{\infty} \sin(1/n^2)$ converges since $\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1.$
- **Ratio & root tests.** Assume $a_n \neq 0$.

  If $\lim_{n\to\infty} \frac{|a_{n+1}|}{a_n} < 1$ or $\lim_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum a_n$ converges (**absolutely**). If $\lim_{n\to\infty} \frac{|a_{n+1}|}{a_n} > 1$ or $\lim_{n\to\infty} |a_n|^{1/n} > 1$, then $\sum a_n$ diverges. These tests are derived by comparison with geometric series. The following are useful in applying the root test: $\lim_{n\to\infty} n^{p/n} = 1$ (any p) and $\lim_{n\to\infty} (n!)^{1/n} = \infty$. More precisely, $\lim_{n\to\infty} \frac{1}{n}(n!)^{1/n} = \frac{1}{e}$.

POWER SERIES

- **Power series**. A power series in x is a sequence of polynomials in x of the form $\sum_{n=0}^{N} a_n x^n$ $(N=0, 1, 2, \ldots)$.

  The power series is denoted $\sum_{n=0}^{\infty} a_n x^n$.

  A power series in $x-c$ (or "centered at c" or "about c") is written $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$.

  Replacing x with a real number q in a power series yields a series of real numbers. A power series **converges at** q if the resulting series of real numbers converges.
- **Interval of convergence**. The set of real numbers at which a power series converges is an interval, called the **interval of convergence**, or a point. If the power series is centered at c, this set is either (i) $(-\infty,\infty)$; (ii) $(c-R, c+R)$ for some $R>0$, possibly together with one or both endpoints; or (iii) the point c alone. In case (ii), R is called the **radius of convergence** of the power series, which may be ∞ and 0 for cases (i) and (iii), respectively. Convergence is absolute for $|x-c|<R$. You can often determine a radius of convergence by solving the inequality that puts the ratio (or root) test limit less than 1. E.g., for $\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$, $\lim_{n\to\infty} \frac{|x|^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{|x|^n} = \frac{|x|}{2} < 1 \Rightarrow |x|<2$, which, with the ratio test, shows that the radius of convergence is 2.
- **Geometric power series**. A power series determines a function on its interval of convergence:

  $x \mapsto f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$. One says the series converges to the function. The series $\sum_{n=0}^{\infty} x^n$, i.e., the sequence of polynomials $\sum_{n=0}^{N} x^n = 1+x+x^2+\ldots+x^N = \frac{1-x^{N+1}}{1-x} (x \neq 1)$, converges for x in the interval $(-1,1)$ to $1/(1-x)$ and diverges otherwise. That is, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} (|x|<1)$. Other geometric series may be identified through this basic one.

  E.g., $\sum_{n=0}^{\infty} 2\cdot 3^{-n} x^n = 2\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \frac{2x}{3}\cdot\frac{1}{1-x/3}$, for $|x/3|<1$. The interval of convergence is $(-3,3)$.
- **Calculus of power series.** Consider a function given by a power series centered at c with radius of convergence R:

  $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$.

  Such a function is **differentiable** on $(c-R, c+R)$, and its derivative there is $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$.

  The differentiated series has radius of convergence R, but may diverge at an endpoint where the original converged. Such a function is **integrable** on $(c-R, c+R)$, and its integral vanishing at c is:

  $\int_c^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} (|x-c|<R)$.

  The integrated series has radius of convergence R, and may converge at an endpoint where the original diverged.

  E.g., $\frac{1}{1+x} = 1-x+x^2\cdots$ implies $\frac{1}{1+x} = 1-x+x^2\cdots$.

  The initial (geometric) series converges on $(-1,1)$, and the integrated series converges on $(1,-1)$. The integration says $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ for $|x|<1$; a remainder argument (see below) implies equality for $x=1$.
- **Taylor and MacLaurin series.** The Taylor series about c of an infinitely differentiable function f is

  $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$.

  If $c=0$, it is also called a MacLaurin series. The Taylor series at x may converge without converging to $f(x)$. It converges to $f(x)$ if the remainder in Taylor's formula, $R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\cdot(x-c)^{(n+1)}$ (ξ between c and x, ξ varying with x and n), approaches 0 as $n\to\infty$. E.g., the remainders at $x=1$ for the MacLaurin polynomials of $\ln(1+x)$ (in **Taylor's formula** above) satisfy

  $|R_n(1)| = \frac{1}{(n+1)(1+\xi)^{n+1}} \le \frac{1}{n+1} \to 0,$

  so $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
- **Computing Taylor series**. If $R>0$ and $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n (|x-c|<R)$, then the coefficients are necessarily the Taylor coefficients: $a_n = f^{(n)}(c)/n!$. This means Taylor series may be found other than by directly computing coefficients. Differentiating the geometric series gives $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n \quad (|x|<1)$.
- **Basic MacLaurin series**:

  $\frac{1}{1-x} = 1+x+x^2+\ldots = \sum_{n=0}^{\infty} x^n \quad (|x|<1)$

  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{n=1}^{\infty} (-1)^{(n+1)}\frac{x^n}{n} \quad (-1<x\le 1)$

  $\ln\frac{1+x}{1-x} = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} \quad (|x|<1)$

  $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (|x|\le 1)$

  The following hold for all real x:

  $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

  $\cos x = 1 - x + \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- **Binomial series**. For $p \neq 0$, and for $|x|<1$,

  $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty}\binom{p}{k}x^k$.

  The **binomial coefficients** are $\binom{p}{0}=1$, $\binom{p}{1}=p$, $\binom{p}{2}=\frac{p(p-1)}{2}$, and ("$p$ choose k") $\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$

  If p is a positive integer, $\binom{p}{k}=0$ for $k>p$.

U.S. $4.95 CAN. $7.50 **Author:** Gerald Harnet, PhD
Customer Hotline # 1.800.230.9522
ISBN-13: 978-142320413-8
ISBN-10: 142320413-1